On deriving flux freezing in magnetohydrodynamics by direct differentiation

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Abstract. The magnetic flux freezing theorem is a basic principle of ideal magnetohydrodynamics (MHD), a commonly used approximation to describe the aspects of astrophysical and laboratory plasmas. The theorem states that the magnetic flux—the integral of magnetic field penetrating a surface—is conserved in time as that surface is distorted in in time by fluid motions. Pedagogues of MHD commonly derive flux freezing without showing how to take the material derivative of a general flux integral and/or assuming a vanishing field divergence from the outset. Here I avoid these shortcomings and derive flux freezing by direct differentiation, explicitly using a Jacobian to transform between the evolving field-penetrating surface at different times. The approach is instructive for its generality and helps elucidate the role of magnetic monopoles in breaking flux freezing. The paucity of appearances of this derivation in standard MHD texts suggests that its pedagogic value is underappreciated.

1. Introduction

Magnetohydrodynamics (MHD) is the simplest generalization of hydrodynamics for a sufficiently ionized collisional plasma [1, 2, 3]. The relative motion between positive and negative charge carriers creates currents which can sustain magnetic fields and electric fields are generated by induction. Charge separation and plasma oscillations are assumed to occur on small enough spatial and temporal scales that the plasma is considered to be neutral on macroscopic scales of interest. As in hydrodynamics, a high rate of particle interactions ensures that deviations from Maxwellian velocity distributions are small.

Non-relativistic MHD limit has been a mainstay of theoretical astrophysics since most of the material inside stars and between them is composed of non-relativistic magnetized plasma and is commonly treated in the MHD approximation. The limit is also widely used in approximating the dynamics and stability of fusion device plasmas. Students in astrophysics and plasma physics are typically exposed to MHD either in advanced undergraduate or graduate courses.

The solution of physical problems in MHD requires equations for mass conservation equation, momentum conservation, energy evolution equation and the magnetic induction equation. The latter is the subject of the present paper and is given in CGS units by [3]

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) - \nabla \times (\nu_M \nabla \times \mathbf{B}), \tag{1}$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{V}$ is the plasma velocity, and $\nu_M \equiv \frac{\eta c^2}{4\pi}$ is the magnetic diffusivity in terms of the resistivity η and speed of light c. The form of Eqn (1) is identical to that of vorticity evolution in incompressible gravitational hydrodynamics if $\mathbf{B}$ is replaced by vorticity $\boldsymbol{\omega} \equiv \nabla \times \mathbf{V}$ and ν_M is replaced by the kinematic viscosity ν.

Eq (1) is derived by starting with Faraday's law $\frac{\partial \mathbf{B}}{\partial t} = c \nabla \times \mathbf{E}$, where $\mathbf{E}$ is the electric field. Eliminating $\mathbf{E}$ in terms of $\mathbf{B}$ and $\mathbf{J}$ is then accomplished by use of Ohm's law, $\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J}$, where $\mathbf{J}$ is the current density. Use of the non-relativistic Ampères law $\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{J}$ then relates $\mathbf{J}$ and $\mathbf{B}$. The aforementioned Ohm's law is derived by subtracting the separate momentum density equations for positive and negative charge carriers. The resistive term arises in Ohm's law when there are finite but small deviations from Maxwellian distributions of the charge carriers. That these deviations are assumed to be weak highlights that the relevance of MHD when many collisions between charged particles occurs over dynamical times of interest. Collisionless plasmas have more complicated Ohm's laws.

The induction equation of *ideal MHD* corresponds to Eq. (1) when the ν_M term is neglected. The ratio of magnitudes of the second term to the third term in Eq. (1) can be approximated by the magnetic Reynolds number $R_M \equiv \frac{VL}{\nu_M}$, where V is the velocity magnitude and L is the characteristic gradient scale of velocity or magnetic field. When $R_M >> 1$, the resistive term is often ignored and ideal MHD assumed. Actually, many astrophysical plasmas are turbulent which implies a spectrum of eddies of different scales and energies. Such flows always have a microphysical dissipation

scale at which $R_M = 1$ so in practice one must think about R_M as a scale dependent quantity. Subtleties involved with applying the limit of ideal MHD to turbulent flows are discussed in [4] and [5]. For the laminar (non-turbulent) case, the limit $R_M >> 1$ leads more straightforwardly to the ideal MHD approximation and that is the focus of this paper.

The ideal MHD limit is indeed the limit for which "magnetic flux freezing" or Alfvén's theorem holds. This theorem (analogous to the Kelvin circulation theorem of ideal hydrodynamics [6, 7] in which vorticity flux is frozen) states that the magnetic flux through a material surface is conserved even as velocity flows distort that surface in time. Mathematically this means

$$\frac{D\Phi_B}{Dt} = 0, \tag{2}$$

where D/Dt indicates the material or Lagrangian derivative $\frac{\partial}{\partial t} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla$ for space and time dependent flow velocity $\mathbf{V}(\mathbf{x}, t)$ and magnetic flux $\Phi_B \equiv \int \mathbf{B} \cdot d\mathbf{S}$, where the integral represents a surface integral over an open surface. The rest of this paper addresses derivations of Eq. (2).

In section 2, I derive Eq. (2) by a formal material derivative of the flux integral. In section 3, I compare this derivation to other derivations commonly found in standard texts. In section 4, I address why the derivation of section 2 facilitates a better physical understanding of the role magnetic monopoles play in violating flux freezing compared to the other derivations discussed in section 3 . I conclude in section 4.

2. Material derivative of the flux and derivation of flux freezing

Consider an open surface within a plasma at $t = 0$ through which magnetic field lines penetrate. The surface has a differential area $dS_0 = d\sigma_1 d\sigma_2$, where σ_1 and σ_2 define local Cartesian coordinates. Via the distorting action of a smoothly varying time and space dependent velocity flow $\mathbf{V}$, this surface evolves to a new surface with differential $d\mathbf{S}$ at time $t \geq 0$. To compute $D\Phi_B/Dt$ we must then account for the fact that both $\mathbf{B}$ and $d\mathbf{S}$ can depend on space and time.

The time-evolved surface measure $d\mathbf{S}$ can be related to the initial surface measure by a Jacobian transformation. With this in mind, let us define a general flux $\Phi_Q \equiv \int \mathbf{Q} \cdot d\mathbf{S}$ for an arbitrary vector $\mathbf{Q}$, not necessarily divergence free. Then

$$\frac{D\Phi_Q}{Dt} = \frac{D}{Dt} \int \mathbf{Q} \cdot d\mathbf{S} = \frac{D}{Dt} \int \mathbf{Q} \cdot \boldsymbol{\mathcal{J}} dS_0, \tag{3}$$

where $d\mathbf{S}$ is the differential surface area vector at an arbitrary $t \geq 0$ and $dS_0 = ||d\mathbf{S}_0||$ is the scalar differential surface area of the surface at $t = 0$. Here $\boldsymbol{\mathcal{J}}$ is the Jacobian vector relating $d\mathbf{S}$ to a coordinate transformation of $d\mathbf{S}_0$, and has components

$$\mathcal{J}_q = \epsilon_{qrs} \frac{\partial x_r}{\partial \sigma_1} \frac{\partial x_s}{\partial \sigma_2}, \tag{4}$$

where ϵ_{qrs} is the Levi-Civita symbol and x_1, x_2, x_3 are the local Cartesian coordinates of the evolving surface element $d\mathbf{S}$. Three coordinates are required for the evolving surface as the surface normal can evolve away from its initial direction.

Because the right side of (3) now involves an integral over a fixed surface, we can take the $\frac{D}{Dt}$ inside the integral to obtain

$$\frac{D}{Dt}\int \mathbf{Q} \cdot \boldsymbol{\mathcal{J}} dS_0 = \int \left(\boldsymbol{\mathcal{J}} \cdot \frac{D}{Dt}\mathbf{Q} + \mathbf{Q} \cdot \frac{D\boldsymbol{\mathcal{J}}}{Dt} \right) dS_0. \tag{5}$$

We now need an expression for $D\boldsymbol{\mathcal{J}}/Dt$. Changing the indices q, r, s in (4) to k, i, j and taking the material derivative gives

$$\frac{D\mathcal{J}_k}{Dt} = \epsilon_{kij} \left[\frac{\partial(Dx_i/Dt)}{\partial\sigma_1}\frac{\partial x_j}{\partial\sigma_2} + \frac{\partial x_i}{\partial\sigma_1}\frac{\partial(Dx_j/Dt)}{\partial\sigma_2} \right]. \tag{6}$$

Using $\frac{Dx_j}{Dt} = V_j$; $\frac{Dx_i}{Dt} = V_i$; $\frac{\partial V_i}{\partial\sigma_1} = \frac{\partial V_i}{\partial x_m}\frac{\partial x_m}{\partial\sigma_1}$; and $\frac{\partial V_j}{\partial\sigma_2} = \frac{\partial V_j}{\partial x_m}\frac{\partial x_m}{\partial\sigma_2}$ in Eq. (6) then gives

$$\frac{D\mathcal{J}_k}{Dt} = \epsilon_{kij} \left[\frac{\partial V_i}{\partial x_m}\frac{\partial x_m}{\partial\sigma_1}\frac{\partial x_j}{\partial\sigma_2} + \frac{\partial x_i}{\partial\sigma_1}\frac{\partial V_j}{\partial x_m}\frac{\partial x_m}{\partial\sigma_2} \right]. \tag{7}$$

Since $\epsilon_{kij} = -\epsilon_{kji}$, we can interchange indices i and j in (7) to obtain

$$\begin{aligned} \frac{D\mathcal{J}_k}{Dt} &= \epsilon_{kij}\frac{\partial V_i}{\partial x_m} \left[\frac{\partial x_m}{\partial\sigma_1}\frac{\partial x_j}{\partial\sigma_2} - \frac{\partial x_j}{\partial\sigma_1}\frac{\partial x_m}{\partial\sigma_2} \right] = \epsilon_{kij}\frac{\partial V_i}{\partial x_m}\epsilon_{mjq}\mathcal{J}_q \\ &= \mathcal{J}_k\nabla \cdot \mathbf{V} - \mathcal{J}_i\frac{\partial V_i}{\partial x_k}, \end{aligned} \tag{8}$$

where the second equality follows from using (4) multiplied by ϵ_{kmn} and the third equality follows from using $\epsilon_{kij}\epsilon_{mjq} = \delta_{qk}\delta_{mi} - \delta_{qi}\delta_{mk}$.

Using (8), along with the definition D/Dt and the fact that $(\mathbf{V} \cdot \nabla\mathbf{Q})_i = \mathbf{V} \cdot \nabla Q_i$ in Cartesian coordinates, we now obtain for (5)

$$\frac{D}{Dt}\int \mathbf{Q} \cdot \boldsymbol{\mathcal{J}} dS_0 = \int \left(\frac{\partial Q_i}{\partial t} + \mathbf{V} \cdot \nabla Q_i + Q_i\nabla \cdot \mathbf{V} - \mathbf{Q} \cdot \nabla V_i \right) \mathcal{J}_i dS_0. \tag{9}$$

Using $\mathcal{J}_i dS_0 = dS_i$ and the vector identity

$$\nabla\times(\nabla\times\mathbf{Q}) = \mathbf{Q} \cdot \nabla\mathbf{V} - \mathbf{V} \cdot \nabla\mathbf{Q} - \mathbf{Q}\nabla \cdot \mathbf{V} + \mathbf{V}\nabla \cdot \mathbf{Q}, \tag{10}$$

Eq. (9) becomes

$$\frac{D}{Dt}\int \mathbf{Q} \cdot d\mathbf{S} = \int \left(\frac{\partial\mathbf{Q}}{\partial t} - \nabla\times(\mathbf{V} \times \mathbf{Q}) + \mathbf{V}\nabla \cdot \mathbf{Q} \right) \cdot d\mathbf{S}. \tag{11}$$

Thus if $\frac{\partial\mathbf{Q}}{\partial t} = \nabla\times(\mathbf{V} \times \mathbf{Q})$ and $\nabla \cdot \mathbf{Q} = 0$, then $\frac{D}{Dt}\int \mathbf{Q} \cdot d\mathbf{S} = 0$. Replacing $\mathbf{Q}$ with $\mathbf{B}$ we have the proof of flux freezing by direct differentiation.

The derivation above is just the surface integral analogue of that used to prove Reynolds transport theorem. The latter describes the evolution of a scalar volume integral over a material volume that evolves in time from a velocity flow [8, 9, 10].

3. Comparison to other derivations

Like the derivation above, the approaches in [11] and [12] do proceed by direct differentiation of the flux integral and separately compute $D\mathbf{B}/dt$ and $Dd\mathbf{S}/dt$ by considering the infinitesimal evolution of these quantities. However these approaches make no explicit mention of the Jacobian so the connection to the basic method of

surface integral transformations is not made explicit. They also assume $\nabla \cdot \mathbf{B} = 0$ and do not provide a physical interpretation of why $\nabla \cdot \mathbf{B} \neq 0$ can violate flux freezing.

The derivation of the Kelvin circulation theorem in Ref. [7] also proceeds by direct differentiation, but uses Stokes' theorem first so that the derivative is taken on the line integral of velocity. The fact that vorticity is the curl of the velocity automatically ensures that vorticity is divergence-free. Also, if the same approach were applied to magnetic flux then the mathematical analogue to the velocity in the proof is the vector potential which itself is not a gauge invariant quantity. The derivation of section 1 avoids use of vector potential for the case that $\mathbf{Q} = \mathbf{B}$ and carries the divergence term to the very end.

Most noteworthy is that in addition to starting with $\nabla \cdot \mathbf{B} = 0$, common MHD presentations [1, 2, 3, 13, 14] do not cleanly show how to calculate the material derivative of the flux integral. Instead these approaches are characterized by the following: A bounded open surface C in the plasma is considered to evolve in a small time δt to a new surface C' by the action of differentiable velocity flows. Because $\nabla \cdot \mathbf{B} = 0$, Gauss' theorem tells us that the total integrated flux at time $t + \delta t$ through the closed surface formed by C, C' and the quasi-cylindrical "side" connecting C and C' is zero. Mathematically, this means

$$\int \nabla \cdot \mathbf{B} \, dV = 0 = - \int_C \mathbf{B}(\mathbf{r}, t+\delta t) \cdot d\mathbf{S} + \int_{C'} \mathbf{B}(\mathbf{r}, t+\delta t) \cdot d\mathbf{S} + \int_{side} \mathbf{B}(\mathbf{r}, t+\delta t) \cdot d\mathbf{S}. \quad (12)$$

The last term is

$$\int_{side} \mathbf{B}(\mathbf{r}, t+\delta t) \cdot d\mathbf{S} = \int_C \mathbf{B}(\mathbf{r}, t+\delta t) \cdot (d\mathbf{l} \times \mathbf{V} \delta t) = \int_C \delta t (\mathbf{V} \times \mathbf{B}(\mathbf{r}, t+\delta t)) \cdot d\mathbf{l}, \quad (13)$$

where $d\mathbf{l}$ is the line element around the boundary of surface C. The differential change in magnetic flux through C as it evoles from C to C' is

$$\delta \Phi_B = \int_{C'} B(\mathbf{r}, t + \delta t) \cdot d\mathbf{S} - \int_C B(\mathbf{r}, t) \cdot d\mathbf{S}. \quad (14)$$

Using (13) and (14) to replace the last and penultimate terms of (12) respectively, gives

$$\frac{\delta \Phi}{\delta t} = \int_C \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S} + \int_C (\mathbf{V} \times \mathbf{B}(\mathbf{r}, t)) \cdot d\mathbf{l} = \int_C \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S} + \int_C \nabla \times (\mathbf{V} \times \mathbf{B}(\mathbf{r}, t)) \cdot d\mathbf{S}, \quad (15)$$

where the last equality follows from use of Green's theorem and the limit of small δt has allowed replacement of $\frac{B(\mathbf{r}, t+\delta t) - B(\mathbf{r}, t)}{\delta t}$ by $\frac{\partial \mathbf{B}}{\partial t}$. Finally, by writing $\delta \Phi / \delta t = D\Phi / Dt$ and using Eq. (1) for $\nu_M = 0$, Eq. (2) obtains.

By comparison to the method of the previous section, derivations along the lines of those which follow Eq. (12)-(15) do not as lucidly separate of the material time derivative of the integrand from that of the measure. In addition, since the starting point assumes $\nabla \cdot \mathbf{B} = 0$, the reader is not provided with the opportunity to understand why $\nabla \cdot \mathbf{B} \neq 0$ can violate flux freezing.

4. Seeing the role of $\nabla \cdot \mathbf{B}$

The need for clarity on why $\nabla \cdot \mathbf{Q} \neq 0$ can violate flux freezing is further evidenced by the ambiguity of the derivation of the Kelvin vorticity theorem of Ref. [15].

There (in contrast to the approach of [7]) it is implied that any vector $\mathbf{Q}$ satisfying $\frac{\partial \mathbf{Q}}{\partial t} - \nabla \times (\mathbf{V} \times \mathbf{Q}) = 0$ obeys flux freezing and that the property that $\mathbf{Q}$ is the curl of some other function (i.e. that $\nabla \cdot \mathbf{Q} = 0$) is unnecessary to prove flux conservation. But this is incorrect, and seemingly results in Ref. [15] from an ambiguity in distinguishing the partial and material derivative (compare Eq. 44 of Ref. [15] to Eqn. 5 above). As seen in Eq. (11) above, a finite divergence term would violate flux freezing even if the former condition is satisfied. Note that the last term in Eq. (11) containing the divergence actually cancels the hidden divergence term within the penultimate term since $-\nabla \times (\mathbf{V} \times \mathbf{Q})$ includes a term $-\mathbf{V} \nabla \cdot \mathbf{Q}$ when expanded with vector identities. However, the ideal MHD magnetic induction equation does not involve $\nabla \cdot \mathbf{B} = 0$ in its derivation. Thus for $\mathbf{Q} = \mathbf{B}$, the first two terms on the right of (11) cancel, leaving the divergence term whose physical meaning I now discuss.

The divergence term of Eq. (11) represents the net advection of field line divergence through the evolving surface. For $\mathbf{Q} = \mathbf{B}$, a finite $\nabla \cdot \mathbf{B} = 4\pi \rho_m$ would imply the existence of magnetic monopoles of magnetic charge density ρ_m by analogy to Gauss' law for electric charge. The magnetic field lines emanating from a magnetic monopole have a net magnetic flux through any spherical surface surrounding the monopole. To see that a net advection of magnetic monopoles through a surface would change the magnetic flux through that surface, first consider the contribution form a single monopole of positive magnetic charge which has all field lines directed radially outward. As the monopole approaches the surface from one side and passes through to the other, the sign of its contribution to the flux through that surface changes. An advection of a net density of monopoles of one sign through the surface would then by extension also change the flux through the surface with time.

Note that a $\nabla \cdot \mathbf{B}$ term in the flux evolution equation need not be the only consequence to MHD in a hypothetical plasma of arbitrarily large magnetic monopole densities. In the same way that we derive the standard Ohm's law for MHD by subtracting electron and ion momentum density equations, we would also have to derive a magnetic Ohm's law by subtracting positive and negative magnetic monopole charge density equations. The two Ohm's laws would be coupled. Further study of magnetic monopoles is beyond the scope of the current paper.

5. Conclusion

Commonly used derivations of magnetic flux freezing tiptoe around showing how to take the material derivative of a general flux integral. In addition, derivations which begin with $\nabla \cdot \mathbf{B} = 0$ from the outset do not provide the opportunity for a physical understanding of why $\nabla \cdot \mathbf{B} \neq 0$ could violate flux freezing. The direct differentiation method of section 1 overcomes both of these shortcomings.

Acknowledgments

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